SIZE DIRECTION GAMES OVER THE REAL LINE. II

BY

GADI MORAN

ABSTRACT

Variants of two basic infinite games of perfect information are studied. A notion of continuous strategy for the player S (Size) is shown to be related to a notion of convergence norm for sequences of reals. With each such norm, a variant of each of the basic games is associated in which the size player has to see that each play obeys the norm. Restriction to choose only rational numbers is also imposed on S. Some games are completely solved, and in this case S has a winning strategy iff his set includes a perfect subset, and D has a winning strategy iff S's set is at most denumerable. Some other games, in which S has to choose only rationals and obey a norm, induce a hierarchy structure on the class of nowhere dense perfect sets, that is embedded cofinally in the lattice of infinite sequences of integers modulo finite differences.

0. Introduction and results

Infinite two person games of perfect information over the real line were first introduced and studied by Banach and Mazur, and were subsequently modified and studied by various authors. See [2] for further references. Here we study a family of variants of two such games that were studied in [1].

Let X be a subset of R, the real line. We associate with X two basic infinite games of perfect information, $\Gamma^D(X)$ and $\Gamma^S(X)$, played by the two players S (Size) and D (Direction) as follows [1]: S chooses a real number s_0. Once s_n is determined, $s_{n+1} = s_n + \varepsilon_n x_n$, where x_n is a positive number chosen by S, and $\varepsilon_n \in \{-1, 1\}$ is chosen by D. In the game $\Gamma^D(X)$, D chooses ε_n first, and then S chooses x_n. In $\Gamma^S(X)$ roles are interchanged: S chooses x_n first, and then D chooses ε_n. The sequence $s = \langle s_n : n < \omega \rangle$ is a *play* in the game. S wins if s is convergent, and its limit — the *outcome* of the play — belongs to X.

It is shown in [1] that $\Gamma^D(X)$ is a win for D if X is discrete, and a win for S

Received December 16, 1971

otherwise. It is also shown there that $\Gamma^S(X)$ is a win for D if X is at most denumerable, and that if $\Gamma^S(X)$ is a win for S then X includes a perfect subset.

This article includes a completion of the solution of $\Gamma^S(X)$. We show that $\Gamma^S(X)$ is a win for S if X includes a perfect subset (Corollary 4.9), and that if it is a win for D then X is at most denumerable (Theorem 5.8).

Our main purpose is to study some games obtained from the basic games by imposing certain restrictions on the way that S is allowed to play. We consider restrictions of two kinds: size restrictions and continuity restrictions.

By a size restriction we mean a restriction imposed on S to play so that the elements s_n constructed through a play belong always to a given countable dense set Q. Q should satisfy the condition that for every $s \in Q$ and every $x \in R$ $s + x \in Q$ iff $s - x \in Q$. This is needed in order to enable the play in Γ^S to be carried on. We conveniently take Q to be the set of the rationals, but all our results hold for any other set satisfying the above requirement.

The notion of a continuity restriction arises by an analysis of the notion of a continuous strategy for S in these games. It is natural to say that a strategy σ is *continuous* if the outcome of every play where σ is used can be approximated by a finite segment of the play. More precisely, σ is continuous if for every play $\langle s_n : n \langle \omega \rangle$ where σ is used and every $\varepsilon > 0$, a natural number N exists so that if $\langle s'_n : n \langle \omega \rangle$ is another play where σ is used, and if $s_n = s'_n$ for $n < N$, then $|\lim s_n - \lim s'_n| < \varepsilon$. σ is *uniformly continuous* if the number N depends only on ε.

A strategy for S in our basic games is actually a labeling σ of the full binary tree 2^* by reals, that satisfies certain conditions (Def. 2.0), one of them being that the sequence of reals obtained by restricting σ to any path in 2^* is convergent. Thus, with each strategy σ for S is associated a mapping $\tilde{\sigma}$ of the set of all infinite paths in 2^* — or, equivalently, of Cantor's discontinuum C — into R: $\tilde{\sigma}$ maps every path to the limit of the sequence labeling it. $\tilde{\sigma}$, considered as a mapping of the topological space C into R suggests another possible way to classify a strategy as a continuous one, namely: σ is a continuous strategy iff $\tilde{\sigma}$ is continuous. It turns out that in our games, continuity in any of the above senses and uniform continuity do all coincide.

By a *convergence norm* (con) we mean any sequence $\boldsymbol{a} = \langle a_n : n < \omega \rangle$ of positive real numbers such that $a_{n+1} \leqq a_n$ and $\lim a_n = 0$. We say that a sequence of real numbers $\boldsymbol{s} = \langle s_n : n < \omega \rangle$ *obeys* the con $\boldsymbol{a}$ if for every $n, m, m' \in \omega$, if

$n \leqq m, m'$ then $|s_m - s_{m'}| < a_n$. We show next that a strategy σ for S is continuous iff all the plays consistent with σ obey some fixed con $\boldsymbol{a}$. The continuity restrictions we impose on S are nothing but such convergence norms.

We consider the following games:

${}_{\boldsymbol{a}}\Gamma^D(X)$ (${}_{\boldsymbol{a}}\Gamma^S(X)$) is the same as $\Gamma^D(X)$ ($\Gamma^S(X)$) except that S loses any play that does not obey the con $\boldsymbol{a}$.

$\tilde{\Gamma}^D(X)$ ($\tilde{\Gamma}^S(X)$) is played as follows. At the beginning, S chooses a con $\boldsymbol{a}$. Then he chooses s_0 and a play $\langle s_n : n < \omega \rangle$ is then constructed as in $\Gamma^D(X)$ ($\Gamma^S(X)$). S wins iff the play $\boldsymbol{s}$ obeys $\boldsymbol{a}$ and its outcome belongs to X. D wins otherwise.

Each of the above games gives rise to another one, obtained from it by further restricting S by the size restriction, i.e., by making S lose any play $\boldsymbol{s} = \langle s_n : n < \omega \rangle$ such that for some n, $s_n \notin Q$. This new game is denoted as its ancestor, except that a subscript Q is added. Thus, for example, ${}_{\boldsymbol{a}}\Gamma^S_Q(X)$ is the same as $\Gamma^S(X)$ except that S is restricted to move so that always $s_n \in Q$, and loses any play that does not obey $\boldsymbol{a}$.

We say that a set X is *a win for* $S(D)$ in one of these games if $S(D)$ has a winning strategy in the corresponding game. Each of our games partitions the power set of R into three: the class of all sets X such that X is a win for S, the class of all sets X such that X is a win for D, and the class of nondetermined sets. We call two games equivalent if they define the same partition. Once the partition is shown to coincide with a familiar one, we say that *the game is solved.*

Our results are summarized in Table 1. We use the following abbreviations:

$S(X)$: X is a win for S

$D(X)$: X is a win for D

$N(X)$: X is at most denumerable

$\bar{P}(X)$: Every perfect set P has a perfect subset P' such that $P' \cap X = \emptyset$.

$I(X)$: X is of the first category.

$P(X)$: X includes a perfect subset.

It is clear that $\bar{P}(X)$ implies: not $P(X)$ and $P(R - X)$.

Because of obvious implications, only the boxes marked with an asterisk need to be proved. Whenever those appear in the text, their consequences are stated in subsequent corollaries.

TABLE 1

The game \ X	$P(X)$	$N(X)$	$S(X)$	$D(X)$
$\tilde{\Gamma}^D$	$S(X)$	$D(X)$ (*5)	$P(X)$ (*7)	$N(X)$
$\tilde{\Gamma}^D_Q$	$S(X)$	$D(X)$	$P(X)$	$N(X)$
${}_a\Gamma^D$	$S(X)$	$D(X)$	$P(X)$	$N(X)$
${}_a\Gamma^D_Q$	$S(X)$ (*1)	$D(X)$	$P(X)$	$N(X)$ (*9)
Γ^S	$S(X)$	$D(X)$ (*6)	$P(X)$ (*8)	$N(X)$
Γ^S_Q	$S(X)$	$D(X)$	$P(X)$	$N(X)$ (*10)
$\tilde{\Gamma}^S$	$S(X)$	$D(X)$	$P(X)$	$\bar{P}(X)$ & $I(X)$
$\tilde{\Gamma}^S_Q$	$S(X)$ (*2)	$D(X)$	$P(X)$	$\bar{P}(X)$ & $I(X)$ (*11)
${}_a\Gamma^S$	$S(X)$ (*3)	$D(X)$	$P(X)$	$I(X)$
${}_a\Gamma^S_Q$	Depends on (*4) a and X	$D(X)$	$P(X)$	$I(X)$ (*12)

Remarks and open problems

Let ZF denote Zermolo-Frankel set theory without the axiom of choice AC, ZFC is ZF + AC, and CH is the continuum hypothesis.

1. All the results stated in Table 1 are theorems of ZF.

2. Γ^D is equivalent to Γ^D_Q and in both games, X is a win for S if it has an accumulation point, and a win for C otherwise [1].

3. All the games completely solved are equivalent, except those mentioned in Remark 2. These are the games $\tilde{\Gamma}^D$, $\tilde{\Gamma}^D_Q$, ${}_a\Gamma^D$, ${}_a\Gamma^D_Q$, Γ^S and Γ^S_Q. In each of them, X is a win for S iff X includes a perfect subset, and is a win for D iff X is at most denumerable. The first half of the last statement, holds for all the games listed in Table 1 except ${}_a\Gamma^S_Q$ and if X is denumerable, then X is a win for D in all these games. If neither X nor its complement include a perfect set, then X is non-determined in any of the games of Table 1 (Corollary 5.15; compare [2, Th. 1]).

4. Solovay [4] showed that, assuming the existence of a strongly inaccessible cardinal, a model of ZF exists in which every uncountable set of reals includes

a perfect subset. It follows that one cannot prove in ZF that there is an uncountable X such that X is a win for D in either of $\tilde{\Gamma}^S$, $\tilde{\Gamma}^S_Q$ or ${}^S_a\tilde{\Gamma}$. On the other hand, it follows from ZFC + CH that there is an uncountable X such that $\tilde{\Gamma}^S_Q(X)$ is a win for D [3]. The existence of such an X for $\tilde{\Gamma}^S$ or ${}_a\Gamma^S$ is an open problem (see Remark 6).

5. We shall prove the following two facts concerning ${}_a\Gamma^S_Q$:

(a) Let X be a perfect set. Then there is a con $\boldsymbol{a}$ such that ${}_a\Gamma^S_Q(X)$ is a win for S (Theorem 4.8).

(b) Let $\boldsymbol{a}$ be a con and X a perfect set. Then there is a perfect subset X' of X such that ${}_a\Gamma^S_Q(X')$ is a win for D (Theorem 5.5).

Let us denote by $(\boldsymbol{a}, X) \equiv (\boldsymbol{a}', X')$ the following statement: ${}_a\Gamma^S_Q(X)$ is a win for S iff ${}_{a'}\Gamma^S_Q(X')$ and ${}_a\Gamma^S_Q(X)$ is a win for D iff ${}_{a'}\Gamma^S_Q(X')$ is a win for D.

Define an equivalence relation E on the set of cons by: $\boldsymbol{a}\, E\, \boldsymbol{a}'$ iff for every X, $(\boldsymbol{a},X) \equiv (\boldsymbol{a}', X)$. Denote by $[\boldsymbol{a}]$ the E-equivalence class of $\boldsymbol{a}$. The set of all E-equivalence classes is partially ordered by: $[\boldsymbol{a}] \leqq_1 [\boldsymbol{a}']$ iff for every X, if ${}_{a'}\Gamma^S_Q(X)$ is a win for D then ${}_a\Gamma^S_Q(X)$ is a win for D.

For cons $\boldsymbol{a}, \boldsymbol{a}'$ we put $\boldsymbol{a} \leqq_2 \boldsymbol{a}'$ iff for some $n, n' \in \omega$, $a_{n+k} \leqq a'_{n'+k}$ for all $k \in \omega$. Let E' be the equivalence relation defined by: $\boldsymbol{a}\, E' \boldsymbol{a}'$ iff $\boldsymbol{a} \leqq_2 \boldsymbol{a}'$ and $\boldsymbol{a}' \leqq_2 \boldsymbol{a}$, and denote by $['\boldsymbol{a}]$ the E'-equivalence class of $\boldsymbol{a}$. $\leqq_2$ naturally partially orders the set of all E'-equivalence classes.

One can show that E' refines E, i.e., for every con $\boldsymbol{a}$, $['\boldsymbol{a}] \subseteq [\boldsymbol{a}]$, and that for any two cons $\boldsymbol{a}, \boldsymbol{b}$, $['\boldsymbol{a}] \leqq_2 ['\boldsymbol{b}]$ implies $[\boldsymbol{a}] \leqq_1 [\boldsymbol{b}]$.

PROBLEM 1. Are E and E' equal ?

Dually, define an equivalence relation F on the power set of R by: XFX' iff for every con $\boldsymbol{a}$, $(\boldsymbol{a}, X) \equiv (\boldsymbol{a}, X')$. Let $[X]$ denote the F-equivalence class of X. Partially order the set of all F-equivalence classes by: $[X] \leqq_3 [X']$ iff for every con $\boldsymbol{a}$, if ${}_a\Gamma^S_Q(X)$ is a win for D then also ${}_a\Gamma^S_Q(X')$ is a win for D. This order induces a hierarchy structure on the power set of R. By (a) and (b), this hierarchy is non-trivial. Moreover, one can show that the lattice of sequences of positive integers modulo finite differences is embeddable cofinally in the set of E'-equivalence classes partially ordered by $\leqq_2$. It follows (using AC) that our hierarchy, even when restricted to perfect sets, embodies ascending chains of length at least ω_1.

Observe that $[X] \leqq_3 [X']$ means that X' is "thiner" than X (in fact, $X' \subseteq X$ implies $[X] \leqq_3 [X']$). It is easy to see that every set X with a nonempty interior is a win for S in ${}_a\Gamma^S_Q(X)$ for every con $\boldsymbol{a}$; thus, these sets do all belong to the $\leqq_3$-minimal F-equivalence class.

PROBLEM 2. Is there a set X of positive Lebesgue measure such that ${}_\alpha\Gamma^S_Q(X)$ is a win for D ?

Of course, it is sufficient to consider only perfect nowhere dense sets X.

6. Concerning the role of D in the games ${}_a\Gamma^S$ and $\tilde{\Gamma}^S$, we know that X is a win for D for every denumerable X. (This follows from [1, Th. 2]. A direct simple proof is actually presented here, Lemma 5.1.)

PROBLEM 3. Assume that ${}_a\Gamma^S(X)$, or even that $\tilde{\Gamma}^S(X)$, is a win for D. Does it follow that X is at most denumerable ?

The answer is not known even if CH and AC are assumed (compare Remark 3). All we can say at present is that, under each of the above assumptions, X is of the first category and does not include a perfect subset (hence, of measure zero, if measurable). If $\tilde{\Gamma}^S(X)$ is a win for D then, moreover, every perfect set has a perfect subset disjoint from X. If ${}_a\Gamma^S(X)$ is a win for D, all we can say is that every perfect set has a subset of the power of the continuum disjoint from X (Corollary 5.14).

The paper is organized as follows. In §2, the various notions of a continuous strategy for S are shown to be equivalent. The relation between such a strategy and a con is established. All statements included in the third column of Table 1 follow easily, an exception being (*8), which is [1, Corollary 2].

In §3, we study some auxiliary open games played by D and S. We obtain some technical results that are used in the sequel.

Section 4 is dedicated mainly to prove the assertions of the first column of Table 1, except (*4). (*1), (*2), and (*3) are respectively Lemma 4.4, Theorem 4.8 and Theorem 4.7. The proof of (*3) is based on an idea of Jan Mycielski. (*2) is actually (a) of Remark 5.

In §5, the other statements are proved. One exception is (*6), which is [1, Th. 2]. Another proof for it is given in [3]. (*4) follows from the more precise statement (b) of Remark 5 and from (*2). (b) is actually Theorem 5.5.

This paper is based on portions from the author's dissertation, prepared at the Hebrew University. I take this opportunity to thank my advisor, Azriel Levy, for his kind supervision. I am indebted to Jan Mycielski for many helpful remarks, and to Saharon Shelah for interesting suggestions and conversations.

1. Notation

ω, Q, Q^+, R and R^+ denote respectively the set of natural numbers, the set of rationals, the set of positive rationals, the set of real numbers and the set of positive reals. An ordinal number is identified with the set of his predecessor. If A, B are sets, $|A|$ denotes the cardinality of A, and $A - B$ is the set theoretical difference of A and B.

If A is a set, $n \leqq \omega$, then A^n is the set of all sequences of order type n of elements from A. $A^* = \cup_{n<\omega} A^n$. If $\xi \in A^n$, we write $l\xi = n$. If $\xi, \eta \in A^*$, then $\xi \cdot \eta$ denotes the concatenation of ξ and η. $\xi \prec \zeta$ if for some $\eta \in A^*$, $\zeta = \xi \cdot \eta$. For $\alpha \in A^n$ $(n \leqq \omega)$, $\bar{\alpha} \in \Pi_{k \leqq n} A^k$ is defined by: $\bar{\alpha}(k)$ is the restriction of α to k. Thus, if $k < \omega$, $k \leqq l\alpha$, then $\bar{\alpha}(k) = \langle \alpha(0), \cdots, \alpha(k-1) \rangle$. $P \subset A^*$ is *a path* if P is a maximal subset of A^* linearly ordered by $\prec$. A^ω is considered also as a topological space. The topology is the one generated by the sets U_ξ, $\xi \in A^*$, where $U_\xi = \{\alpha \in A^\omega: \bar{\alpha}(l\xi) = \xi\}$.

If $A \subseteq R$, then $\bar{A}$ is the closure of A, $\operatorname{int} A$ is its interior, CON A is the convex hull of A and mA is its Lebesgue measure. If $A, B \subset R$, $A < B$ means $a < b$ for all $a \in A$, $b \in B$, $A + B = \{a + b: a \in A, b \in B\}$. $A < b$, $A + b$ stands for $A < \{b\}$, $A + \{b\}$, etc. The distance between A and B is denoted by $\mathrm{d}(A, B)$, and defined by $\mathrm{d}(A, B) = \inf\{|a - b|: a \in A, b \in B\}$. We write $\mathrm{d}(s, A)$ for $\mathrm{d}(\{s\}, A)$.

A *convergence norm* (*con*) is a sequence $\boldsymbol{a} = \langle a_n: n < \omega \rangle$ of positive numbers such that $a_{n+1} \leqq a_n$ and $\lim a_n = 0$. $\boldsymbol{s} = \langle s_n: n < \omega \rangle \in R^\omega$ *obeys* the con $\boldsymbol{a}$ if for every $n, m, m' \in \omega$, $n \leqq m, m'$ implies $|s_m - s_{m'}| < a_n$.

2. Continuous strategies for S

DEFINITION 2.0. A *strategy for S in* Γ^D (Γ^S) is a mapping σ: $2^* \to R$ that satisfies:

(0) $\langle \sigma(\bar{\alpha}(n)): n < \omega \rangle$ is a convergent sequence for every $\alpha \in 2^\omega$.

(1) $\sigma(\xi \cdot \langle 0 \rangle) < \sigma(\xi) < \sigma(\xi \cdot \langle 1 \rangle)$ for every $\xi \in 2^*$.

((2) $\sigma(\xi) - \sigma(\xi \cdot \langle 0 \rangle) = \sigma(\xi \cdot \langle 1 \rangle) - \sigma(\xi)$ for evey $\xi \in 2^*$.)

A strategy for S in $\Gamma^D_Q(\Gamma^S_Q)$ is similarly defined, except that now σ is a function from 2^* into Q.

Let σ be a strategy for S. Then $\tilde{\sigma}$: $2^\omega \to R$, $L(\sigma) \subset R$ and $P(\sigma) \subset R^\omega$ are defined as follows:

$$\tilde{\sigma}(\alpha) = \lim \sigma(\bar{\alpha}(n))$$
$$L(\sigma) = \{\tilde{\sigma}(\alpha): \alpha \in 2^\omega\}$$
$$P(\sigma) = \{\langle \sigma(\bar{\alpha}(n)): n < \omega \rangle: \alpha \in 2^\omega\}.$$

$\tilde{\sigma}$ is *the outcome function*, $L(\sigma)$ is *the set of outcomes* and $P(\sigma)$ is the *set of plays* associated with σ.

σ is a *winning strategy* in $\Gamma^D(X)$, $\Gamma^S(X)$, $\Gamma^D_Q(X)$ or $\Gamma^S_Q(X)$ if it is a strategy for S in the corresponding game and $L(\sigma) \subseteq X$.

Let $\sigma: 2^* \to R$ satisfy (0). Then $\tilde{\sigma}$ is well defined. We say that σ is *continuous* if $\tilde{\sigma}: 2^\omega \to R$ is continuous.

Let $\sigma: 2^* \to R$ be any function. We say that σ is *uniformly continuous* (*u.c.*) if the following holds: For every $\varepsilon > 0$, there exists a natural number n such that for every $\xi \in 2^n$, ζ, $\zeta' \in 2^*$ $\left|\sigma(\xi \cdot \zeta') - \sigma(\xi \cdot \zeta)\right| < \varepsilon$.

It is clear that if σ is u.c., then (0) is satisfied, so $\tilde{\sigma}(\alpha)$ is well-defined for every $\alpha \in 2^\omega$.

Note that every strategy for S in $\Gamma^S(\Gamma^S_Q)$ is also a strategy for S in $\Gamma^D(\Gamma^D_Q)$.

LEMMA 2.1. *Let σ be a strategy for S in $\Gamma^D(\Gamma^D_Q)$. Then σ is continuous if and only if σ is uniformly continuous.*

PROOF. i) Assume that σ is continuous, and let $\varepsilon > 0$ be given. We shal find $n \in \omega$ as required in the definition of u.c.

For $\alpha \in 2^\omega$ put $V_\alpha = \{\alpha' \in 2^\omega: \left|\tilde{\sigma}(\alpha') - \tilde{\sigma}(\alpha)\right| < \frac{1}{2}\varepsilon\}$. Let $n_\alpha \in \omega$ satisfy: $U^{\alpha}_{\bar{\alpha}(n_\alpha)} \subseteq V_\alpha$. Since $\alpha \in U_{\bar{\alpha}(n_\alpha)}$, $\{U_{\bar{\alpha}(n_\alpha)}: \alpha \in 2^\omega\}$ is an open cover of 2^ω. Thus there are $\alpha_0, \cdots, \alpha_{k-1}$ so that

$$(*) \qquad 2^\omega = \bigcup_{0 \leqq i < k} U_{\alpha_i(n_{\alpha_i})}.$$

Put $n = \max\{n_{\alpha_i}: 0 \leqq i < k\}$. We claim that this n satisfies the requirements. Assume that $\xi \in 2^n$, ζ, $\zeta' \in 2^*$, and $\sigma(\xi \cdot \zeta') - \sigma(\xi \cdot \zeta) \geqq \varepsilon$. Define $\beta_0, \beta_1 \in 2^\omega$ by:

$$\bar{\beta}_0(l\xi + l\zeta) = \xi \cdot \zeta, \quad \beta_0(m) = 0 \text{ for } m \geqq l\xi + l\zeta$$

$$\bar{\beta}_1(l\xi + l\zeta') = \xi \cdot \zeta', \quad \beta_0(m) = 1 \text{ for } m \geqq l\xi + l\zeta'.$$

By (1) of Definition 2.0, we have for $m \geqq l\xi + l\zeta$, $l\xi + l\zeta'$:

$$\tilde{\sigma}(\beta_1) - \tilde{\sigma}(\beta_0) > \sigma(\bar{\beta}_1(m)) - \sigma(\bar{\beta}_0(m)) \geqq \sigma(\xi \cdot \zeta) - \sigma(\xi \cdot \zeta) \geqq \varepsilon.$$

On the other hand, since $\xi \in 2^n$, we must have, by (*), $\bar{\alpha}_i(n_{\alpha_i}) \prec \xi$ for some i, $0 \leqq i < k$. Thus, $U_\xi \subseteq U_{\alpha_i(n_{\alpha_i})} \subseteq V_{\alpha_i}$. But $\beta_0, \beta_1 \in U_\xi$, so $\tilde{\sigma}(\beta_1) - \tilde{\sigma}(\beta_0) \leqq$ $\leqq \left|\tilde{\sigma}(\beta_1) - \tilde{\sigma}(\alpha_i)\right| + \left|\tilde{\sigma}(\alpha_i) - \tilde{\sigma}(\beta_0)\right| < \varepsilon$, a contradiction.

ii) Assume that σ is uniformly continuous. Let $\alpha \in 2^\omega$, $\varepsilon > 0$ be given. Pick n so that for $\xi \in 2^n$, $\zeta, \zeta' \in 2^*$, $\left|\sigma(\xi \cdot \zeta') - \sigma(\xi \cdot \zeta)\right| < \frac{1}{2}\varepsilon$. Then for every $\alpha \in U_{\bar{\alpha}(n)}$ we have:

$$\left|\tilde{\sigma}(\alpha') - \tilde{\sigma}(\alpha)\right| = \left|\lim \sigma(\bar{\alpha}'(m)) - \lim \sigma(\bar{\alpha}(m))\right|$$
$$\leqq \lim_{n \leqq m} \left|\sigma(\bar{\alpha}'(m)) - \sigma(\bar{\alpha}(m))\right| \leqq \tfrac{1}{2}\varepsilon < \varepsilon. \qquad \square$$

Observe that we have shown, in fact, that if any $\sigma: 2^* \to R$ is uniformly continuous, then $\tilde{\sigma}$ is continuous. The other direction need not be true, and (0), (1) of Definition 2.0 are needed for it. (0) alone is not sufficient as the following example shows. Define $\sigma: 2^* \to R$ by $\sigma(\langle 0 \rangle^n \cdot \langle 1 \rangle) = 1$, $n \in \omega$, and $\sigma(\xi) = 0$; otherwise, $\tilde{\sigma}(\alpha)$ is defined and equals 0 for every $\alpha \in 2^\omega$, so $\tilde{\sigma}$ is continuous but it is clear that σ is not uniformly continuous.

LEMMA 2.2. *Let σ be a continuous strategy for S in Γ^D. Then $L(\sigma)$ is a compact perfect set.*

PROOF. Since $\tilde{\sigma}$ is continuous, $L(\sigma)$ is compact. We have to show only that $L(\sigma)$ has no isolated points. Let $\tilde{\sigma}(\alpha)$ be any member of $L(\sigma)$, and ε any positive number. By Lemma 2.1, we may assume that σ is u.c. Let n satisfy: if $\xi \in 2^n$, $\zeta, \zeta' \in 2^*$ then $\left|\sigma(\xi \cdot \zeta) - \sigma(\xi \cdot \zeta')\right| < \tfrac{1}{2}\varepsilon$. Define $\beta_0, \beta_1 \in 2^\omega$ by $\bar{\beta}_0(n) = \bar{\beta}_1(n) = \bar{\alpha}(n)$, $\beta_0(m) = 0$, $\beta_1(m) = 1$ for $m \geqq n$. Then $\tilde{\sigma}(\beta_0) < \tilde{\sigma}(\beta_1)$ (by (1) of Definition 2.0) and they both belong to $(\tilde{\sigma}(\alpha) - \varepsilon, \tilde{\sigma}(\alpha) + \varepsilon)$. $\square$

COROLLARY 2.3. *If S has a continuous winning strategy in any of $\Gamma^D(X)$, $\Gamma^D_Q(X)$, $\Gamma^S(X)$, $\Gamma^S_Q(X)$, then X includes a perfect subset.*

The relation between continuous strategies for S and cons is stated in the next theorem.

THEOREM 2.4. *Let σ be a strategy for S in Γ^D. Then σ is continuous if and only if there is a con $\boldsymbol{a}$ such that every $\boldsymbol{s}$ in $P(\sigma)$ obeys $\boldsymbol{a}$.*

PROOF. i) Assume that σ is a continuous strategy for S in Γ^D. By Lemma 2.1, σ is u.c. and hence the following definition makes sense: p_k is the least number $n \in \omega$ such that for every $\xi \in 2^n$ and for every $\zeta, \zeta' \in 2^*$, $\left|\sigma(\zeta) - \sigma(\zeta')\right| < 1/(k+1)$. Define now: $n_0 = p_0$, $n_{k+1} = \max\{n_k + 1, p_{k+1}\}$, $b_0 = 1 + \max\{\left|s' - s''\right| : s', s'' \in L(\sigma)\}$, $b_k = 1/k$ for $k > 0$. Lastly, define the con $\boldsymbol{a} = \langle a_n : n < \omega \rangle$ as follows:

$$a_n = b_0 \qquad 0 \leqq n < n_0$$
$$a_n = b_{k+1} \qquad n_k \leqq n < n_{k+1}.$$

It is clear that every $\boldsymbol{s} \in P(\sigma)$ obeys $\boldsymbol{a}$.

ii) Assume that $\boldsymbol{a} = \langle a_n : n < \omega \rangle$ is a con such that every $\boldsymbol{s} \in P(\sigma)$ obeys $\boldsymbol{a}$.

Let $\varepsilon > 0, \delta \in 2^\omega$ be given. Let $n \in \omega$ satisfy $2a_n < \varepsilon$. We claim that $U_{\alpha(n)} \subseteq \tilde{\sigma}^{-1}(\tilde{\sigma}(\alpha) - \varepsilon, \tilde{\sigma}(\alpha) + \varepsilon)$ and hence, σ is continuous. Indeed, assume that $\alpha \in U_{\alpha(n)}$. Then, since both $\langle \sigma(\bar{\alpha}(m)) : m < \omega \rangle$ and $\langle \sigma(\bar{\alpha}'(m)) : m < \omega \rangle$ obey $\boldsymbol{a}$, we have, for $m \geqq n$ (recall that $\bar{\alpha}(n) = \bar{\alpha}'(n)$):

$$\left|\sigma(\bar{\alpha}'(m)) - \sigma(\bar{\alpha}(m))\right| \leqq \left|\sigma(\bar{\alpha}'(m)) - \sigma(\bar{\alpha}'(n))\right| + \left|\sigma(\bar{\alpha}(n)) - \sigma(\bar{\alpha}(m))\right| < 2a_n,$$

hence

$$\left|\tilde{\sigma}(\alpha') - \tilde{\sigma}(\alpha)\right| = \lim \left|\sigma(\bar{\alpha}'(m)) - \sigma(\bar{\alpha}(m))\right| \leqq 2a_n < \varepsilon.$$

□

COROLLARY 2.5. *Let σ be a strategy for S in either of the games $\Gamma^D, \Gamma^D_Q, \Gamma^S, \Gamma^S_Q$. Then σ is continuous iff for some con $\boldsymbol{a}$, every element of $P(\sigma)$ obeys $\boldsymbol{a}$.*

COROLLARY 2.5 justifies the somewhat loose statement, that $\tilde{\Gamma}^D(X)$, $\tilde{\Gamma}^D_Q(X)$, $\tilde{\Gamma}^S(X)$, $\tilde{\Gamma}^S_Q(X)$ are obtained respectively from $\Gamma^D(X)$, $\Gamma^D_Q(X)$, $\Gamma^S(X)$, $\Gamma^S_Q(X)$ by imposing on S the restriction to use only continuous strategies.

3. The games $G(X;s;n)$ and $G_Q(X;s;n)$

We study here a family of auxiliary open games which are the main tool in our subsequent study of the variants of Γ^S.

Let X be a set of real numbers, $s \in R$ and $n \leqq \omega$. The game $G(X;s;n)$ ($G_Q(X;s;n)$) is played by S and D as follows. S chooses $x_0 \in R^+$ ($x_0 \in Q^+$) and then D chooses $\varepsilon_0 \in \{-1, 1\}$; S chooses $x_1 \in R^+$ ($x_1 \in Q^+$) and then D chooses $\varepsilon_1 \in \{-1, 1\}$, and so on. Put $s_k = s + \sum_{i<k} \varepsilon_i x_i$. S wins if for some $0 \leqq k \leqq n$ $k < \omega$, $s_k \in X$. D wins otherwise.

A neat solution of the games $G(X;s;n)$, $G_Q(X;s;n)$ is formulated in terms of two operations on the set of reals that we define now.

DEFINITION 3.0. Let $X \subseteq R$. Then:

$$TX = \{\tfrac{1}{2}(x' + x''): x', x'' \in X\}$$

$$T_Q X = \{\tfrac{1}{2}(x' + x''): x', x'' \in X \text{ and } x' - x'' \in Q\}.$$

$T^n X$, $T^n_Q X$ are defined for $0 \leqq n \leqq \omega$ by:

$$T^0 X = T^0_Q X = X.$$

$$T^{n+1} X = T\,T^n X. \quad T^{n+1}_Q X = T_Q T^n_Q X.$$

$$T^\omega X = \bigcup_{n<\omega} T^n X. \quad T^\omega_Q X = \bigcup_{n<\omega} T^n_Q X.$$

$\rho(s)$, $\rho_Q(s)$ are defined for $s \in R$ by:

$$\rho(s) = \min\{n: s \in T^n X\} \text{ if } s \in T^\omega X$$

$$\rho_Q(s) = \min\{n: s \in T^n_Q X\} \text{ if } s \in T^\omega_Q X$$

$$\rho(s) = \omega(\rho_Q(s) = \omega) \text{ if } s \notin T^\omega X\,(s \notin T^\omega_Q X).$$

From Definition 3.0 one easily derives:

Proposition 3.1. (i) *If* $\rho(s) = n + 1$ $(\rho_Q(s) = n + 1)$ *then for some* $x \in R^+(x \in Q^+)$, $\max\{\rho(s+x), \rho(s-x)\} \leqq n$ $(\max\{\rho_Q(s+x), \rho_Q(s-x)\} \leqq n)$.

(ii) *If* $\rho(s) \geqq n + 1$ $(\rho_Q(s) \geqq n + 1)$ *then for every* $x \in R^+$ $(x \in Q^+)$ *there is an* $\varepsilon \in \{-1, 1\}$ *such that* $\rho(s + \varepsilon x) \geqq n$ $(\rho_Q(s + \varepsilon x) \geqq n)$.

(iii) *If* $\rho(s) = \omega(\rho_Q(s) = \omega)$ *then for every* $x \in R^+$ $(x \in Q^+)$ *there is an* $\varepsilon \in \{-1, 1\}$ *such that* $\rho(s + \varepsilon x) = \omega$ $(\rho_Q(s + \varepsilon x) = \omega)$.

Theorem 3.2. $G(X; s; n)$ $(G_Q(X; s; n))$ *is a win for S if* $s \in T^n X$ $(s \in T^n_Q X)$ *and is a win for D otherwise.*

Proof. Optimal strategies for S and D are suggested by Proposition 3.1. We shall give the proof for $G(X; s; n)$; the proof for $G_Q(X; s; n)$ is its "Q-analog".

Assume that $s \in T^n X$. If $n = 0$ then $s \in X$ and hence $s_0 \in X$ always, so the game $G(X; s; 0)$ is a win for S. If $0 < n$, we may assume $n < \omega$, and by Proposition 3.1 (i), S can choose $x \in R^+$ so that $\rho(s + x)$, $\rho(s - x) < n$. Continuing this way, he enters X in at most n moves.

Assume that $s \notin T^n X$. If $n < \omega$, Proposition 3.1 (ii) tells us that for any $x \in R^+$ chosen by S, D may find $\varepsilon \in \{-1, 1\}$ so that $\rho(s + \varepsilon x) \geqq n$. Continuing this way, D may avoid X during the first n moves, and hence $G(X; s; n)$ is a win for D. If $n = \omega$, Proposition 3.1 (iii) tells us that D can ensure that $s_k \notin T^\omega X$ for all $k < \omega$, and thus, again, $G(X; s; n)$ is a win for D. □

Two special kinds of X will interest us later; first, X is a union of two intervals, and second, X is finite.

Assume first that $X = X_0 \cup (X_0 + d)$ for some $X_0 \subseteq R$, $d \in R$. Then

$$T^n X \supseteq \bigcup_{k=0}^{2^n} X_0 + \frac{k}{2^n} d$$

and if $d \in Q$, then $T^n_Q X \supseteq \bigcup_{k=0}^{2^n} (X_0 + k/2^n\, d)$. Hence, if g is an open interval, $mg = a > 0$, $d \in Q^+$ and $X = g \cup (g + d)$, then for n such that $2^n a > d$ we have:

$$T_Q^n X = \mathrm{CON}\, X.$$

Using Theorem 3.2, we get:

LEMMA 3.3. *Let g, g' be nonempty open intervals. Then there is a natural number n such that for every $s \in \mathrm{CON}\,X$, $G_Q(g \cup g'; s; n)$ is a win for S. The least such n is denoted $n(g, g')$.*

Next we want to show that if X is finite and X' is obtained from X by moving each of its elements by no more than δ, then also T^nX is obtained from T^nX by moving each of its elements by no more than δ.

Let y_n be a real variable for each $n < \omega$. Define an increasing chain $\mathscr{F}^n$ of sets of real valued functions by $\mathscr{F}^0 = \{y_n : n < \omega\}$, $\mathscr{F}^{n+1} = \{\frac{1}{2}(f' + f'') : f', f'' \in \mathscr{F}^n\}$ and $\mathscr{F}^\omega = \bigcup_{n<\omega} \mathscr{F}^n$. For $f \in \mathscr{F}^\omega$, $\sup(f) \subset \omega$ is defined by: $\sup(y_n) = \{n\}$, $\sup(\frac{1}{2}(f' + f'')) = \sup(f') \cup \sup('')$. If $X = \{x_0, \cdots, x_{k-1}\} \subset R$ is a finite indexed set and $f \in \mathscr{F}^\omega$, $\sup(f) \subseteq \{0, \cdots, k-1\} = k$, then $f(x_0, \cdots, x_{k-1})$ is the real number obtained by substituting x_i for y_i in f. Put $\mathscr{F}_k^n = \mathscr{F}^n \cap \{f : \sup(f) \subseteq k\}$.

One proves by induction

PROPOSITION 3.4. i) *If $f \in \mathscr{F}^n$ then $0 < |\sup(f)| \leqq 2^n$.*

(ii) *If $f \in \mathscr{F}^n$ and $\sup(f) = \{i_0, \cdots, i_{k-1}\}$, then there are positive natural numbers $m_0, \cdots, m_{k-1}$ such that $\sum_{0 \leqq i < k} m_i = 2^n$ and*

$$f = \frac{m_0}{2^n} y_{i0} + \cdots + \frac{m_{k-1}}{2^n} y_{i_{k-1}}$$

(iii) *$\mathscr{F}_k^n$ is finite for $n, k < \omega$. In fact, $|\mathscr{F}_k^n| = \binom{2^n + k - 1}{k - 1}$.*

(iv) *If $x = \{x_0, \cdots, x_{k-1}\} \subset R$ is a finite indexed set, $n < \omega$, then*

$$T^nX = \{f(x_0, \cdots, x_{k-1}) : f \in \mathscr{F}_k^n\}.$$

LEMMA 3.5. *Let $X = \{x_0, \cdots, x_{k-1}\}$ be a finite indexed set, $\delta > 0$, $n < \omega$ be given. Then*

$$T^n \bigcup_{x \in X} [x - \delta, x + \delta] = \bigcup_{x \in T^nX} [x - \delta, x + \delta].$$

PROOF. i) Assume that $s' \in T^n \bigcup_{x \in X}[x - \delta, x + \delta]$. By Proposition 3.4 (iv), there is an $f \in \mathscr{F}_k^n$, $x_i' \in \bigcup_{x \in X}[x - \delta, x + \delta]$, $0 \leqq i < k$, such that $s' = f(x_0', \cdots, x_{k-1}')$. Moreover, f and x_i' can be chosen so, that $x_i' \in [x_i - \delta, x_i + \delta]$, $0 \leqq i < k$. Assume it, and put $s = f(x_0, \cdots, x_{k-1})$. By Proposition 3.4 (ii), there are $a_0, \cdots, a_{k-1}$ so that $0 \leqq a_i$, $\sum_{0 \leqq i < k} a_i = 1$, and $f = a_0 y_0 + \cdots + a_{k-1} y_{k-1}$. It follows that $\left|s - s'\right| = \left|\Sigma_{0 \leqq i < k} a_i(x_i - x_i')\right| \leqq \delta$. $\Sigma_{0 \leqq i < k} a_i = \delta$. But $s \in T^n X$, so $s' \in \bigcup_{x \in T^n X}[x - \delta, x + \delta]$.

ii) Assume that $s' \in [x - \delta, x + \delta]$ where $x \in T^n X$. Then $s' = x + \delta'$ for some $-\delta \leqq \delta' \leqq \delta$. Let $f \in \mathscr{F}_k^n$ satisfy $x = f(x_0, \cdots, x_{k-1})$, and assume that $f = a_0 y_0 + \cdots + a_{k-1} y_{k-1}$, $0 \leqq a_i$, $\Sigma_{0 \leqq i < k} a_i = 1$. Let $x_i' \in [x_i - \delta, x_i + \delta]$ be defined by $x_i' = x_i + \delta'$. Then $f(x_0', \cdots, x_{k-1}') \in T^n \bigcup_{0 \leqq i < k}[x_i - \delta, x_i + \delta]$ and $f(x_0', \cdots, x_{k-1}') = a_0(x_0 + \delta') + \cdots + a_{k-1}(x_{k-1} + \delta') = f(x_0, \cdots, x_{k-1}) + \delta' = x + \delta' = s'$. Thus, $s' \in T^n \bigcup_{x \in X}[x - \delta, x + \delta]$. □

4. The role of S

In §2, it was shown that if X is a win for S in any of our games where he is allowed to use only continuous strategies, then X must include a perfect set. Here we shall show that the converse is also true, an exception being the games ${}_{a}\Gamma_Q^S(X)$, that are treated in the next section.

Definition 4.0. A *binary-interval-system* (*bis*) is a function J defined on 2^* whose values are closed nonempty intervals satisfying:

$$J(\xi \cdot \langle 0 \rangle),\ J(\xi \cdot \langle 1 \rangle) \subset J(\xi)$$

$$J(\xi \cdot \langle 0 \rangle) < J(\xi \cdot \langle 1 \rangle).$$

The *kernel* of a bis J, KJ, is the perfect set defined by:

$$KJ = \bigcap_{n < \omega} \bigcup_{\xi \in 2^n} J(\xi).$$

We say that a bis J *obeys* a con $\boldsymbol{a} = \langle a_n : n < \omega \rangle$ if for all $n \in \omega$, $\xi \in 2^n$:

$$mJ(\xi) < a_n.$$

Observe that if J is a bis, then $\operatorname{int} J(\xi) \neq \varnothing$ for all $\xi \in \omega$, and $d(J(\xi \cdot \langle 0 \rangle), J(\xi \cdot \langle 1 \rangle)) > 0$. Also, if $\boldsymbol{s} = \langle s_n : n < \omega \rangle \in R^\omega$ satisfies, for some $\alpha \in 2^\omega$, $s_n \in J(\bar{\alpha}(n))$, and if J obeys $\boldsymbol{a}$, then $\boldsymbol{s}$ obeys $\boldsymbol{a}$.

Definition 4.1. Let $X \subseteq R$. Then $H(X)$ denotes the set of all real numbers s such that for every $\varepsilon > 0$, $(s - \varepsilon, s) \cap X \neq \varnothing$ and $(s, s + \varepsilon) \cap X \neq \varnothing$.

Lemma 4.2. *Let X be a perfect set.*

i) *If $\boldsymbol{a}$ is a con, J is a bis that obeys $\boldsymbol{a}$, and for every $\xi \in 2^*$ the endpoints of $J(\xi)$ belong to X, then $K \ \subseteq X$.*

ii) *$H(X) \subseteq X$, $H(H(X)) = H(X)$, and $X - H(X)$ is at most denumerable.*

iii) *For every con $\boldsymbol{a}$ there is a bis J such that J obeys $\boldsymbol{a}$ and for every $\xi \in 2^*$, the endpoints of $J(\xi)$ belong to X.*

PROOF. i) Observe that $KJ = \bigcup_{\alpha \in 2^\omega} \bigcap_{n<\omega} J(\bar{\alpha}(n))$. Since J obeys $\boldsymbol{a}$, $\bigcap_{n<\omega} J(\bar{\alpha}(n))$ is a singletone whose only element is a limit of a sequence of elements of X, hence belongs to X.

ii) We note first that $X - H(X)$ is always at most denumerable [2, p. 325]). $H(X) \subseteq X$ whenever X is closed. $H(H(X)) \subseteq H(X)$ is also clearly true of any $X \subseteq R$. Assume that $s \in H(X)$ and $\varepsilon > 0$. We know that $(s-\varepsilon, s) \cap X \neq \emptyset$. But X is perfect, so $(s - \varepsilon, s) \cap X$ has the power of the continuum. It follows that $(s-\varepsilon, s) \cap H(X) \neq \emptyset$. Similarly, $(s, s+\varepsilon) \cap H(X) \neq \emptyset$. Thus $s \in H(H(X))$.

iii) Define by induction on n a_ξ, b_ξ for $\xi \in 2^n$ as follows. $a_\phi, b_\phi \in H(X)$ are arbitrary elements so that $0 < b_\phi - a_\phi < a_0$. Assume that a_ξ, b_ξ are already defined, $0 < b_\xi - a_\xi$, $a_\xi, b_\xi \in H(X)$ for $\xi \in 2^n$. By $H(H(X)) = H(X)$, we can find $a_{\xi \cdot \langle 1 \rangle}$, $b_{\xi \cdot \langle 0 \rangle} \in H(X)$ so that $a_\xi < b_{\xi \cdot \langle 0 \rangle} < a_{\xi \cdot \langle 1 \rangle} < b_\xi$ and $b_{\xi \cdot \langle 0 \rangle} - a_\xi$, $b_\xi - a_{\xi \cdot \langle 1 \rangle} < a_{n+1}$. Put also $a_{\xi \cdot \langle 0 \rangle} = a_\xi$, $b_{\xi \cdot \langle 1 \rangle} = b_\xi$. Clearly $J(\xi) = [a_\xi, b_\xi]$ is a bis satisfying the requirements. □

DEFINITION 4.3. Let J be a bis and assume that $J(\xi) = [a_\xi, b_\xi]$, $\xi \in 2^*$. Then F_J is the function defined on 2^* whose values are open nonempty intervals given by:

$$F_J(\xi) = (b_{\xi \cdot \langle 0 \rangle}, a_{\xi \cdot \langle 1 \rangle}).$$

Thus, $F_J(\xi) \subset J(\xi)$, and $\{F_J(\xi) : \xi \in 2^*\}$ is a disjointed family of open intervals included in CON KJ and whose union is disjoint from KJ.

We shall now prove assertions of the form "if X is perfect, then X is a win for S in a certain game". In each instance we shall actually present a strategy σ such that for a suitable bis J that satisfies the conditions of Lemma 4.2 (i), $KJ = L(\sigma)$ (see Definitions 4.0, 2.0).

LEMMA 4.4. *If X is a perfect set, $\boldsymbol{a}$ a con, then ${}_{\boldsymbol{a}}\Gamma^D_Q(X)$ is a win for S.*

PROOF. Let J be any bis that obeys $\boldsymbol{a}$ such that the endpoints of $J(\xi)$ belong to X for all $\xi \in 2^*$. A winning strategy for S is any function $\sigma : 2^* \to Q$ that satisfies $\sigma(\xi) \in F_J(\xi)$.

□

COROLLARY 4.5. *If X includes a perfect set, then any of ${}_{\boldsymbol{a}}\Gamma^D_Q(X)$, ${}_{\boldsymbol{a}}\Gamma^D(X)$, $\tilde{\Gamma}^D_Q(X)$, $\tilde{\Gamma}^D(X)$ is a win for S.*

The state is different in the case of the variants of Γ^S. The analog of Lemma 4.4 does not hold. If, however, we either release S from the size restriction or keep this restriction but let him choose the con $\boldsymbol{a}$, then he has a winning strategy for every perfect set. The next proposition will be used in the proof of the first of those results.

PROPOSITION 4.6. *Let* $\{t_\zeta : \zeta \in \bigcup_{0 \leqq k \leqq n} 2^k\}$, $\{t'_\zeta : \zeta \in \bigcup_{0 \leqq k \leqq n} 2^k\}$ *be two indexed systems of real numbers satisfying*:

$$t_\zeta = \tfrac{1}{2}(t_{\zeta \cdot \langle 0 \rangle} + t_{\zeta \cdot \langle 1 \rangle}),\ t_\zeta' = \tfrac{1}{2}(t'_{\zeta \cdot \langle 0 \rangle} + t'_{\zeta \cdot \langle 1 \rangle}), \qquad l\zeta < n.$$

If for every $\xi \in 2^n$, $|t'_\xi - t_\xi| < \delta$, *then for every* $\zeta \in \bigcup_{0 \leqq k \leqq n} 2^k$, $|t'_\zeta - t_\zeta| < \delta$.

The proposition is proved by a straight-forward induction on n.

THEOREM 4.7. *If X is a perfect set, $\boldsymbol{a}$ is a con, then ${}_{\boldsymbol{a}}\Gamma^S(X)$ is a win for S.*

PROOF. Let J be a bis satisfying the conditions of Lemma 4.2 (iii). For every $\xi \in 2^*$, $l\xi \leqq n$, define $t_{\xi,n}$ as follows:

$t_{\xi,l\xi}$ is an arbitrary member of $J(\xi)$.

$$(*) \qquad t_{\xi,n+1} = \tfrac{1}{2}(t_{\xi \cdot \langle 0 \rangle, n} + t_{\xi \cdot \langle 1 \rangle, n}).$$

One shows by induction that $t_{\zeta,n} \in J(\zeta)$ for every n, $l\zeta \leqq n$. It follows by Proposition 4.6 that for $l\xi \leqq n \leqq m, m'$, we have (since J obeys $\boldsymbol{a}$):

$$|t_{\xi,m} - t_{\xi,m'}| < a_n.$$

In particular, $\langle t_{\xi,n} : l\xi \leqq n < \omega \rangle$ is a convergent sequence for every $\xi \in 2^*$. Put

$$(**) \qquad \sigma(\xi) = \lim_n t_{\xi,n}.$$

We have to show that (1), (2) of Definition 2.0 are satisfied. Since $t_{\zeta,n} \in J(\zeta)$ for $\zeta \in 2^*$, $l\zeta \leqq n$, we have: $t_{\xi \cdot \langle 1 \rangle n} - t_{\xi \cdot \langle 0 \rangle, n} \geqq d_\xi > 0$ for $n \geqq l\xi + 1$, where $d_\xi = d(J(\xi \cdot \langle 0 \rangle), J(\xi \cdot \langle 1 \rangle))$. Hence also $\sigma(\xi \cdot \langle 1 \rangle) - \sigma(\xi \cdot \langle 0 \rangle) > 0$. Also, by (*), (**) $\sigma(\xi) = \tfrac{1}{2}(\sigma(\xi \cdot \langle 0 \rangle) + \sigma(\xi \cdot \langle 1 \rangle))$. It follows that (1), (2) of Definition 2.0 hold, so σ is a strategy for S in ${}_{\boldsymbol{a}}\Gamma^S$.

□

Observe that the strategy σ given in the last proof is uniquely determined by the bis J, i.e., it is independent of the choice of the elements $t_{\xi,l\xi}$ in $J(\xi)$. It is possible, given a perfect set X, to construct J so that J obeys $\boldsymbol{a}$, $KJ \subseteq X$, but so that the associated σ will take no rational value.

THEOREM 4.8. *If X is a perfect set, then $\tilde{\Gamma}^{\xi}_{Q}(X)$ is a win for S.*

PROOF. Let J be a bis that obeys, say, the con $\langle 1/(n+1): n<\omega\rangle$ and such that $KJ \subseteq X$. For every $\xi \in 2^*$, put

$$n_\xi = n(F_J(\xi \cdot \langle 0\rangle), F_J(\xi \cdot \langle 1\rangle)) \qquad \text{(see Lemma 3.3, Definition 4.3)}$$

$$p_n = \max\{n_\xi : \xi \in 2^n\}$$

$$k_n = \sum_{i \leqq n} p_i .$$

Define a con $\boldsymbol{a}$ by:

$$a_m = \frac{1}{n+1} \text{ for } k_n \leqq m < k_{n+1} .$$

(This means that if $\xi \in 2^n$, then S is able to make at least p_n moves in $J(\xi)$ without violating $\boldsymbol{a}$).

By Lemma 3.3, S has a winning strategy in $G_Q(F_J(\xi \cdot \langle 0\rangle) \cup F_J(\xi \cdot \langle 1\rangle); s; p_n)$ for every $\xi \in 2^*$ and $s \in F_J(\xi) \subset \mathrm{CON}(F_J(\xi \cdot \langle 0\rangle) \cup F_J(\xi \cdot \langle 1\rangle))$. A winning strategy σ for S in $\tilde{\Gamma}^S_Q(X)$ is the following one: S chooses $\boldsymbol{a}$ as his con, and any rational $s_0 = s_\phi \in F_J(\varnothing)$. Then he follows his winning strategy in the game $G_Q(F_J(\langle 0\rangle) \cup F_J(\langle 1\rangle);\, S_\phi;\, p_0)$, and falls after at most k_0 moves into $s_{r_0} = s_{\langle\varepsilon\rangle} \in F_J(\langle\varepsilon\rangle) \cap Q$, $r_0 \leqq k_0$, for some $\varepsilon \in 2$. It is clear that for $0 \leqq j < r_0$, $s_j \in \mathrm{CON}\,(F_J(\langle 0\rangle) \cup F_J(\langle 1\rangle)) \subset J(\phi)$. Assume that for $\xi \in 2^n$, $r_{n-1} \leqq k_{n-1}$, $s_{r_{n-1}} = s_\xi \in F_J(\xi) \cap Q$ is already attained. Then S follows his winning strategy in $G_Q(F_J(\xi \cdot \langle 0\rangle) \cup F_J(\xi \cdot \langle 1\rangle); s_\xi; p_n)$ and reaches $s_{r_n} = s_{\xi\cdot\langle\varepsilon\rangle} \in F_J((\xi \cdot \langle\varepsilon\rangle) \cap Q$ in at most p_n moves which all lie in $J(\xi)$. Thus $r_n \leqq k_n$. It is clear that σ is a winning strategy for S in ${}_{\boldsymbol{a}}\Gamma^S_Q(X)$, since $P(\sigma)$ (see Definition 2.0) includes only sequences that obey $\boldsymbol{a}$. It follows that $\tilde{\Gamma}^S_Q(X)$ is a win for S. □

COROLLARY 4.9. *If X includes a perfect subset, $\boldsymbol{a}$ a con, then any of ${}_{\boldsymbol{a}}\Gamma^S(X)$, $\tilde{\Gamma}^S(X)$, $\tilde{\Gamma}^S_Q(X)$, $\Gamma^S(X)$ is a win for S.*

5. The role of D

We prove here results of two types. Results of the first type state that a certain X is a win for D in a certain restricted game. Results of the second type state that if X is a win for D in some game, then X is not too big. We start with results of the first type. The following notions are useful.

DEFINITION 5.0. If $z, s \in R$ then $\varepsilon \in \{-1, 1\}$ is *a recoil from z at s* if $(s-z)\cdot\varepsilon \geqq 0$. If $\boldsymbol{a} = \langle a_n : n < \omega\rangle$ is a con, $\eta \in R^*$, then $g_{\boldsymbol{a}}(\eta)$ is an open interval, defined by:

$g_a(\varnothing) = R$, and if $l\eta = n$, $s \in R$, then $g_a(\eta \cdot \langle s \rangle) = g_a(\eta) \cap (s - a_n, s + a_n)$. $\eta = \langle s_0, \cdots, s_{n-1} \rangle$ *obeys* $\boldsymbol{a}$ if $s_i \in g_a(\bar{\eta}(i))$ for $i < n$.

Clearly, $s \in R^\omega$ obeys $\boldsymbol{a}$ iff $\bar{s}(n)$ obeys $\boldsymbol{a}$ for all $n \in \omega$.

LEMMA 5.1. *If X is denumerable, then $\tilde{\Gamma}^D(X)$ is a win for D.*

PROOF. Assume that $X = \{z_n : n < \omega\}$. Let $\boldsymbol{a} = \langle a_n : n < \omega \rangle$ be the con chosen by S, and s_0, x_0 his first move. Then D makes ε_0 a recoil from z_0 at s_0. Thus, $s_1 = s_0 + \varepsilon_0 x_0$ satisfies $|s_1 - z_0| = 2b_0 > 0$. Next D picks n_0, the least n such that $a_n \leqq b_0$, and makes ε_i a recoil from z_0 at s_i for $i < n_0$. Assume that $s_0, \cdots, s_{n_{k-1}}$ are already played so that this sequence obeys $\boldsymbol{a}$, and that $x_{n_{k-1}} > 0$ is chosen by S. Let $b_k = \frac{1}{2} x_{n_{k-1}}, n_k, = \min\{n > n_{k-1} : a_n \leqq b_k\}$. Then D makes ε_i a recoil from z_k at s_i for $n_{k-1} \leqq i < n_k$.

Assume that a play $s = \langle s_n : n < \omega \rangle$ is obtained while D played following this strategy, and s obeys $\boldsymbol{a}$. We have for $m \geqq n_k$:

$$s_m \in g_a(s(m)) \subseteq (s_{n_k} - a_{n_k}, s_{n_k} + a_{n_k});$$

hence, $|z_k - s_m| \geqq b_k > 0$, and thus the outcome of s does not equal any member of X.

□

COROLLARY 5.2. *If X is at most denumerable, $\boldsymbol{a}$ is a con, then any of ${}_a\Gamma^D_Q(X)$, ${}_a\Gamma^D(X)$, $\tilde{\Gamma}^D_Q(X)$, $\tilde{\Gamma}^D(X)$ is a win for D.*

We mention here that if S is released from any continuity restriction (i.e., in the games Γ^D, Γ^D_Q) then X is a win for D only if X is discrete [1, Th. 1]. By Theorem 5.6 below, Lemma 5.1 cannot be improved.

Rather surprisingly, the scope of D in the games ${}_a\Gamma^S_Q$ is much wider. The key fact is the following one. Assume that $\{q_n : n < \omega\}$ is any fixed enumeration of Q.

LEMMA 5.3. *Let $\boldsymbol{a} = \langle a_n : n < \omega \rangle$ be a con. For each $n \in \omega$, let F_n be a closed set. Let $\boldsymbol{k} = \langle k_n : n < \omega \rangle$ be a sequence of natural numbers satisfying:*

(*) $\quad a_{k_n} < mg$ *for every component g of $R - F_n$.*

(**) $\quad q_n \notin T^{k_n} F_n$.

Let $X = \bigcap_{n<\omega} F_n$. Then ${}_a\Gamma^S_Q(X)$ is a win for D.

PROOF. Assume that S chooses $s_0 = q_n$. By (**) and Theorem 3.2, D has a winning strategy in $G(F_n; q_n; k_n)$. Following this strategy k_n moves, D makes sure that $s_{k_n} \notin F_n$. Thus, $s_{k_n} \notin g$ for some component g of $R - F_n$. If $(s_{k_n} - a_{k_n}, s_k + a_{k_n})$

has a positive distance from F_n, S already lost the play. Otherwise, by (*), there is a unique $z \in F_n$ such that for $\delta > 0$, $\delta = |s_{k_n} - z| = \min\{|s_{k_n} - z'|: z' \in F_n\}$. D picks $r \in \omega$, $k_n < r$, so that $a_r \leqq \delta$, and makes ε_i a recoil from z at s_i for $k_n \leqq i < r$. Let $\eta \in R^*$ be $\langle s_0, \cdots, s_r \rangle$. Since $g_a(\eta) \subset (s_r - a_r, s_r + a_r) \cap (s_{k_n} - a_{k_n}, s_{k_n} + a_{k_n})$, it follows by (*) that $g_a(\eta)$ has positive distance from F_n (or else it is empty), hence again D won the play.

□

REMARK 5.4. Let ${}_a\Gamma^*(X)$ be the same as ${}_a\Gamma^S(X)$, except that S is restricted to choose $s_0 \in Q$. Lemma 3.3 actually holds for ${}_a\Gamma^*(X)$ instead of ${}_a\Gamma^S_Q(X)$, as we used in the proof only the fact that S starts with an element of Q. The same is true for the next theorem, and this should be compared with Theorem 4.7.

THEOREM 5.5. *Let a be a con, X a perfect set. Then there is a perfect subset X' of X such that ${}_a\Gamma^S_Q(X)$ is a win for D.*

PROOF. By Lemma 5.3 and Lemma 4.2, it is sufficient to construct a bis J such that J obeys $\langle 3^{-n}: n < \omega \rangle$, the endpoints of $J(\xi)$ belong to X for every $\xi \in 2^*$, and a sequence $\langle k_n: n < \omega \rangle$ so that (*) and (**) hold, where $F_n = \bigcup_{\xi \in 2^n} J(\xi)$.

Let $a_\phi, b_\phi \in H(X)$ (see Definition 4.1) satisfy $0 < b_\phi - a_\phi < 1$, $q_0 \notin [a_\phi, b_\phi]$.

Put $J(\emptyset) = [a_\phi, b_\phi]$.

Let $\xi_0, \xi_1, \cdots, \xi_{2^{n-1}-1}$ be the lexicographical ordering of 2^{n-1}. Assume that $J(\xi_i) = [a_{\xi_i}, b_{\xi_i}]$ is already defined so that $a_{\xi_i}, b_{\xi_i} \in H(X)$, $0 \leqq i < j < 2^{n-1}$ implies $J(\xi_i) < J(\xi_j)$, and $mJ(\xi_i) < 3^{-(n-1)}$.

Let $S_n = \{s_i: 0 \leqq i < 2^n\}$ satisfy:

(0) $s_i \in H(X)$

(1) $i < j$ implies $s_i < s_j$

(2) $s_{2i}, s_{2i+1} \in J(\xi_i)$, $0 \leqq i < 2^{n-1}$

(3) S_n is linearly independent over Q.

It follows from (3), by Proposition 3.4, that $Q \cap T^\omega S_n = \emptyset$. Define d_n, k_n, δ_n as follows:

(4) $3d_n = \min\{|s' - s''|: s', s'' \in S_n\}$

(5) $k_n = \min\{m: a_m < d_n\}$

(6) $2\delta_n = \min\{2d_n, \min\{|q_n - s|: s \in T^{k_n}S_n\}\}$.

By (6), $q_n \notin \bigcup_{s \in T^{k_n}S_n} [s - \delta_n, s + \delta_n]$. By Lemma 3.5, we deduce:

(7) $q_n \notin T^{k_n} \bigcup_{s \in S_n} [s - \delta_n, s + \delta_n]$.

Choose now $a_{\xi_i \cdot \langle \varepsilon \rangle}$, $b_{\xi_i \cdot \langle \varepsilon \rangle}$, $0 \leqq i < 2^{n-1}$, $\varepsilon \in 2$ so that:

(8) $a_{\xi_i \cdot \langle \varepsilon \rangle}, b_{\xi_i \cdot \langle \varepsilon \rangle} \in H(X) \cap J(\xi_i)$

(9) $a_{\xi_i \cdot \langle 0 \rangle} < s_{2i} < b_{\xi_i \cdot \langle 0 \rangle} < a_{\xi_i \cdot \langle 1 \rangle} < s_{2i+1} < b_{\xi_i \cdot \langle 1 \rangle}$

(10) $b_{\xi_i \cdot \langle \varepsilon \rangle} - a_{\xi_i \cdot \langle \varepsilon \rangle} \leqq \delta_n$

and

(11) $J(\xi_i \, . \, \langle \varepsilon \rangle) = [a_{\xi_i \cdot \langle \varepsilon \rangle}, b_{\xi_i \cdot \langle \varepsilon \rangle}]$.

It follows from (4), (5), (6), (9), (10) that (*) of Lemma 5.3 holds, and from (7), (9), (10) that (**) holds.

□

We turn now to results of the second type, i.e., we try to answer the following sort of a question:

Given that X is a win for D in a certain game, how thin should X be? All our results refer to games where S is restricted to choose only rationals, and the proofs are mostly based on what we call the method of z-sequences. We define, in each particular case, a notion of z-sequence for each $z \in R$. A z-sequence η is a certain finite sequence of rationals; it is some initial segment of a play where D's winning strategy is used. Then we show that for each $z \in X$, a z-sequence exists. It follows that if A_η is the set of all reals z such that η is a z-sequence, then $X \subseteq \bigcup_\eta A_\eta$, and the union is taken over a countable set. It remains to find out how small A_η should be. In some cases $({}_a\Gamma^D_Q, \Gamma^S_Q)$, we can show that A_η is at most denumerable, and hence so is X. In the other case $({}_a\Gamma^S_Q)$, all we can show is that A_η is nowhere dense, hence X is of the first category.

THEOREM 5.6. *If ${}_a\Gamma^D_Q(X)$ is a win for D, then X is at most denumerable.*

PROOF. Let $\tau: Q^* \to \{-1, 1\}$ be a winning strategy for D in ${}_a\Gamma^D_Q(X)$. $\eta = \langle q_0, \cdots, q_{n-1} \rangle \in Q^*$ is a *τ-sequence* if for $0 < i < n$, $q_i = q_{i-1} + \tau(\bar{\eta}(i)) \cdot |q_i - q_{i-1}|$. $s \in R^\omega$ is a play where D uses τ that obeys $\boldsymbol{a}$ iff for all $n \in \omega$, $\bar{s}(n)$ is a τ-sequence and $s_n \in g_{\boldsymbol{a}}(\bar{s}(n))$ (see Definition 5.0).

Let z be a real number. Define a one place predicate M_z on Q^* as follows. Let $\eta = \langle q_0, \cdots, q_{n-1} \rangle \in Q^n$. Then $M_z(\eta)$ if and only if:

(i) η is a τ-sequence

(ii) $q_i \in g_{\boldsymbol{a}}(\bar{\eta}(i))$ for $i < n$

(iii) $z \in g_{\boldsymbol{a}}(\eta)$.

Observe that $M_z(\eta)$ holds for every $\eta \in Q^0 \cup Q^1$. Assume that $\eta \in Q^n$. η is a z-sequence iff:

(0) $M_z(\eta)$

(1) for every $\delta > 0$ there is a $q \in Q$ such that $|z - q| < \delta$ and $M_z(\eta \cdot \langle q \rangle)$

(2) for every $q \in Q$ such that $|z - q| < a_n$ and $M_z(\eta \cdot \langle q \rangle)$ there is a $\delta > 0$ so that if $M_z(\eta \cdot \langle q, q' \rangle)$ then $|z - q'| \geqq \delta$.

The theorem will follow from the following two claims.

Claim 1. *If no z-sequence exists, then* $z \notin X$.

Claim 2. $A_\eta = \{z \in R : \eta$ *is a z-sequence*$\}$ *is at most denumerable.*

Proof of Claim 1. Define $s \in Q^\omega$ as follows. (0) and (1) of the definition of a z-sequence are true for $\varnothing$. Assume, by induction, that $\bar{s}(n) \in Q^n$ is already defined so that (0) and (1) hold. Since $\bar{s}(n)$ is not a z-sequence, (2) fails. Hence there is a $q_n \in Q$ such that $M_z(\bar{s}(n) \cdot \langle q_n \rangle)$, and also (1) holds for $s(n) \cdot \langle q_n \rangle$. Put $s(n) = q_n$. Since $M_z(\bar{s}(n))$ holds for all n, s is a play that obeys $\boldsymbol{a}$ where D uses τ whose limit is z. Since τ is a winning strategy, it follows that $z \notin X$.

Proof of Claim 2. We shall show that if $l\eta = n$, then every open interval $g' \subseteq g_{\boldsymbol{a}}(\eta)$ with $mg' \leqq a_n$ contains at most two elements of A_η.

Indeed, assume that $z', z, z'' \in A_\eta$, $z' < z < z''$, and $z'' - z' < a_n$. By (1) of the definition applied to z, we can find a $q \in Q$ such that $z' < q < z''$ and $M_z(\eta \cdot \langle q \rangle)$. Since $g_{\boldsymbol{a}}(\eta \cdot \langle q \rangle) = g_{\boldsymbol{a}}(\eta) \cap (q - a_n, q + a_n)$, it follows that also $M_{z'}(\eta \cdot \langle q \rangle)$ and $M_{z''}(\eta \cdot \langle q \rangle)$. By (2) of the definition, applied to z' and z'', there are δ' and δ'' so that for every q such that $M_{z'}(\eta \cdot \langle q, q \rangle)$, $|z - q| \geqq \delta'$ and for every q'' such that $M_{z''}(\eta \cdot \langle q, q'' \rangle)$, $|z'' - q''| \geqq \delta''$. But this is clearly impossible, since if, say, $\tau(\eta \cdot \langle q \rangle) = 1$, any $q' \in g_{\boldsymbol{a}}(\eta \cdot \langle q \rangle)$ such that $|z' - q'| < \delta'$ and $q' < q$ satisfies $M_{z'}(\eta \cdot \langle q, q' \rangle)$.

Corollary 5.7. *If any of* ${}_{\boldsymbol{a}}\Gamma^D_Q(X)$, ${}_{\boldsymbol{a}}\Gamma^D(X)$, $\tilde{\Gamma}^D_Q(X)$, $\tilde{\Gamma}^D(X)$ *is a win for D, then X is at most denumerable.*

Theorem 5.8. *If* $\Gamma^S_Q(X)$ *is a win for D, then X is at most denumerable.*

Proof. Let $\tau : Q^* \times Q^+ \to \{-1, 1\}$ be a winning strategy for D in $\Gamma^S_Q(X)$. For $\bar{\eta} = \langle \eta_0, \cdots, \eta_{n-1} \rangle \in (\breve{Q}^*)^n$, $(\breve{Q}^* = Q^* - \{\varnothing\})$. Put $\hat{\eta} = \eta_0 \cdot \eta_1 \cdots \eta_{n-1}$ and for $\eta' \in Q^*$, $\eta' \neq \varnothing$, $t(\eta')$ denotes the last member of η'. Define $g(\eta)$ for $\eta \in (\breve{Q}^*)^*$ by $g(\varnothing) = R$, and if $\eta \in (\breve{Q}^*)^n$, $\eta' \in \breve{Q}^*$ then $g\langle \eta \cdot (\eta' \rangle) = g(\eta) \cap (t(\eta') - 1/(n+1), t(\eta') + 1/(n+1))$. $\eta' = \langle q_0, \cdots, q_{n-1} \rangle \in Q^*$ is a *τ-sequence* if for $0 < i < n$, $q_i = q_{i-1} + \tau(\bar{\eta}'(i), |q_i - q_{i-1}|) \cdot |q_i - q_{i-1}|$.

Let z be a real number, $\eta = \langle \eta_0, \cdots, \eta_{n-1} \rangle \in (\breve{Q}^*)^*$. Then $M_z(\eta)$ if and only if:

(i) $\hat{\eta}$ is a τ-sequence.

(ii) all entries of η_i belong to $g(\bar{\eta}(i))$, $i < n$

(iii) $z \in g(\eta)$.

$\eta \in (\breve{Q}^*)^n$ is a *z-sequence* iff:

(0) $M_z(\eta)$

(1) for every $\delta > 0$ there is an $\eta' \in \breve{Q}^*$ such that $|t(\eta') - z| < \delta$ and $M_z(\eta \cdot \langle \eta' \rangle)$

(2) for every $\eta' \in \breve{Q}^*$ such that $|t(\eta') - z| < 1(/n+1)$ and $M_z(\eta \cdot \langle \eta' \rangle)$ there is a $\delta > 0$ such that for every $\eta'' \in \breve{Q}^*$, if $M_z(\eta \cdot \langle \eta', \eta'' \rangle)$ then $|t(\eta'') - z| \geqq \delta$.

Claim 1. *If no z-sequence exists then $z \notin X$.*

Proof of Claim 1. Assume that no z-sequence exists. Define $u \in (Q^*)^\omega$ as follows. $\varnothing = \bar{u}(0)$ satisfies (0) and (1). Assume that $\bar{u}(n)$ is already defined so that (0), (1) are satisfied· Since $\bar{u}(n)$ is not a z-sequence, (2) does not hold. Hence we can pick $\eta'_n \in Q^*$ so that $\bar{u}(n) \cdot \langle \eta'_n \rangle$ also satisfy (0), (1) and we put $u(n) = \eta_n$. Let $s \in Q^\omega$ be the unique element that extends $\hat{\bar{u}}(n)$ for all n. But $M_z(\bar{u}(n))$ or all n implies that s is a play in Γ_Q^S where D uses τ, s is convergent and its limit is z. Since τ is a winning strategy, it follows that $z \notin X$.

Claim 2. *$A_\eta = \{z : \eta$ is a z-sequence$\}$ is at most denumerable.*

Proof of Claim 2. Assume that $\eta \in (\breve{Q}^*)^n$. We shall show that no open subinterval g' of $g(\eta)$ with $mg' \leqq 1/(n+1)$ contains more than two members of A_n. Indeed, assume that $z' < z < z''$, $z', z, z'' \in g(\eta) \cap A$, and $z'' - z' < 1/(n+1)$. Apply (1) for z to pick an $\eta' \in Q^*$ such that $M_z(\eta \cdot \langle \eta' \rangle)$ and $z' < t(\eta') < z''$. It follows that also $M_{z'}(\eta \langle \eta' \rangle)$, $M_{z''}(\eta \cdot \langle \eta' \rangle)$. Let δ', δ'' be as guaranteed by (2) for z', z'' respectively. But $t(\eta') \in \mathrm{CON}((z', z' + \delta') \cup (z'' - \delta'', z''))$. By Lemma 3.3, one can find $\eta'' \in Q^*$ such that $t(\eta'') \in (z , z + \delta) \cup (z'' - \delta'', z'')$ and $M_{z'}(\eta \cdot \langle \eta , \eta'' \rangle)$, $M_{z''}(\eta \cdot \langle \eta', \eta'' \rangle)$, a contradiction. □

Corollary 5.9. *If any of $\Gamma_Q^S(X)$, $\Gamma^S(X)$ is a win for D then X is at most denumerable.*

Theorem 5.10. *If ${}_a\Gamma_Q^S(X)$ is a win for D, then X is of the first category.*

Proof. Let $\tau : Q^* \times Q^+ \to \{-1, 1\}$ be a winning strategy for D in ${}_a\Gamma_Q^S(X)$. $\eta = \langle q_0, \cdots, q_{n-1} \rangle \in Q^*$ is a *τ-sequence* if for $0 < i < n$, $q_i = q_{i-1} + \tau(\bar{\eta}(i), |q_i - q_{i-1}|) \cdot |q_i - q_{i-1}|$.

"$M_z(\eta)$" and "η is a z-sequence" are defined as in the proof of Theorem 5.6. So is the proof of

CLAIM 1. *If there is no z-sequence, then* $z \notin X$.

However, by Theorem 5.5, we cannot expect here that A_η will be countable.

CLAIM 2. $A_\eta = \{z : \eta$ *is a z-sequence*$\}$ *is nowhere dense.*

PROOF OF CLAIM 2. Assume that $\eta \in Q^n$. Let $z \in A_\eta$, and let g be any open interval containing z. We have to find an open nonempty interval $g' \subset g$ such that $g' \cap A_\eta = \emptyset$. We may assume that $g \subset g_a(\eta)$ and that $mg \leqq a_n$. Let $\delta_0 > 0$ be defined by $3\delta_0 = d(z, R - g) > 0$. By (1) of the definition of a z-sequence, pick $q \in Q$ such that $|z - q| < \delta_0$ and $M_z(\eta \cdot \langle q \rangle)$. By (2) there is a $\delta > 0$ such that if $M_z(\eta\langle q, q'\rangle)$ then $|z - q'| \geqq \delta$. It is clear that $\delta \leqq \delta_0$. Now define an open interval g' by: $g' = \{q - x: q + x \in (z - \delta, z + \delta)\}$. Then $g' \subset g$ and for every $q' \in g' \cap Q$ we must have $M_z(\eta \cdot \langle q, q' \rangle)$. It follows that $g' \cap A_\eta = \emptyset$.

□

COROLLARY 5.11. *If any of* ${}_a\Gamma^S_Q(X)$, ${}_a\Gamma^S(X)$, $\tilde\Gamma^S_Q(X)$, $\tilde\Gamma^S(X)$ *is a win for D, then X is of the first category.*

By Table 1, we know that if X is a win for D in ${}_a\Gamma^S$, $\tilde\Gamma^S_Q$ or Γ^S, then also X does not include a perfect subset. A little more can be said if $\tilde\Gamma^S_Q(X)$ is a win for D.

THEOREM 5.11. *If* $\tilde\Gamma^S_Q(X)$ *is a win for D, P is a perfect set, then there is a perfect set* $P' \subset P$ *such that* $P' \cap X = \emptyset$.

PROOF. Let τ be a winning strategy for D in $\tilde\Gamma^S_Q(X)$. Let J be any bis such that $KJ \subset P$ that obeys, say, the con $\langle 1/(n+1) : n < \omega \rangle$ (see Definition 4.0). Define a sequence $\langle r_n : n < \omega \rangle$ of natural numbers as follows: $r_0 = 0$, and for $n > 0$: $r_n = \max\{n(F_J(\xi), F_J(\xi')) : \xi, \xi' \in 2^n\}$ (see Lemma 3.3, Definition 4.3). $r_n > 0$ for $n > 0$. Put $k_n = \sum_{i \leqq n} r_i$ and define a con $\boldsymbol{a}^0 = \langle a_m : m < \omega \rangle$ by: $a_m = 1/(n+1)$, $k_n \leqq m \leqq k_{n+1}$.

τ is actually a family $\{\tau_{\boldsymbol{a}} : \boldsymbol{a}$ is a con$\}$ where $\tau_{\boldsymbol{a}} : Q^* \times Q^+ \to \{-1, 1\}$ is a winning strategy for D in ${}_{\boldsymbol{a}}\Gamma^S_Q(X)$. Put $\tau' = \tau_{\boldsymbol{a}^0}$. We shall define by induction a mapping $\phi : 2^* \to 2^*$, $\eta_\xi \in Q^*$, $q_\xi \in Q$, a bis J' such that:

(0) for $\xi, \xi' \in 2^*$, $\xi \prec \xi'$ iff $\phi(\xi) \prec \phi(\xi')$ and $l\phi(\xi) = 2l\xi$

(1) η_ξ is a τ'-sequence (see the proof of Theorem 5.10)

(2) $q_\xi = t(\eta_\xi)$ (see the proof of Theorem 5.8)

(3) η_ξ obeys $\boldsymbol{a}^0$ (see Definition 5.0)

(4) $q_\xi \in F_J(\phi(\xi)) \subset J(\phi(\xi)) = J'(\xi)$ (see Definition 4.0, 4.3).

Let $q_\phi \in F_J)\emptyset) \cap Q$ be an arbitrary element, and put $\eta_\phi = \langle q_\phi \rangle \phi(\emptyset) = \emptyset$, $J'(\emptyset) = J(\emptyset)$.

Assume that $\phi(\xi)$, η_ξ, q_ξ and $J'(\xi)$ are already defined so that (0)–(4) hold for $\xi \in 2^n$. Put

$$V' = F_J(\phi(\xi) \cdot \langle 0,0 \rangle) \cup F_J(\phi(\xi) \cdot \langle 1,1 \rangle)$$

$$V'' = F_J(\phi(\xi) \cdot \langle 0,1 \rangle) \cup F_J(\phi(\xi) \cdot \langle 1,0 \rangle).$$

It is clear that $q_\xi \in \mathrm{CON}\, V'' \subset \mathrm{CON}\, V'$.

By Lemma 3.3, S has winning strategies σ', σ'' in the games $G_Q(V'; q_\xi; r_{n+1})$, $G_Q(V''; q_\xi; r_{n+1})$ respectively. Assuming that η_ξ is obtained already, τ' tells us how D will play against σ', σ'' in these games. We follow τ' in these games at most r_{n+1} moves until two τ'-sequences $\eta_{\xi \cdot \langle 0 \rangle}$, $\eta_{\xi \cdot \langle 1 \rangle}$ that extend η_ξ are obtained so that for $\langle \varepsilon_0, \varepsilon_1 \rangle$, $\langle \varepsilon_0', \varepsilon_1' \rangle \in 2^2$ such that $\langle \varepsilon_0, \varepsilon_1 \rangle$ precedes $\langle \varepsilon_0', \varepsilon_1' \rangle$ lexicographically, we have, for $q_{\xi \cdot \langle \varepsilon \rangle} = t(\eta_{\xi \cdot \langle \varepsilon \rangle})$:

$$q_{\xi \cdot \langle 0 \rangle} \in F_J(\phi(\xi) \cdot \langle \varepsilon_0, \varepsilon_1 \rangle)$$

$$q_{\xi \cdot \langle 1 \rangle} \in F_J(\phi(\xi) \cdot \langle \varepsilon_0', \varepsilon_1' \rangle).$$

We put now $\phi(\xi \cdot \langle 0 \rangle) = \phi(\xi) \cdot \langle \varepsilon_0, \varepsilon_1 \rangle$, $\phi(\xi \cdot \langle 1 \rangle) = \phi(\xi) \cdot \langle \varepsilon_0', \varepsilon_0' \rangle$ and $J'(\xi \cdot \langle \varepsilon \rangle) = J(\phi(\xi \cdot \langle \varepsilon \rangle))$, $\varepsilon \in 2$. It is clear that (0)–(4) are carried over. Now, clearly $P' = KJ \subset KJ \subset P$. But if $s \in P'$, then for some $\alpha \in 2^\omega$, $\{s\} = \bigcap_{n<\omega} J'(\bar{\alpha}(n))$. Let $\mathbf{s} \in Q^\omega$ be the unique element that extends $\eta_{\bar{\alpha}(n)}$ for all n; then $\mathbf{s}$ is a play in ${}_{\mathbf{a}_0}\Gamma_Q^S(X)$ where τ' is used, and its outcome is s. Since τ' is a winning strategy, we conclude that $s \notin X$. So, $P' \cap X = \emptyset$.

□

Observe that we actually proved the following:

THEOREM 5.11′. *Let P be a perfect set. Then there is a con $\mathbf{a}$ such that for every $X \subset R$, if ${}_{\mathbf{a}}\Gamma_Q^S(X)$ is a win for D, then there is a perfect subset P' of P such that $P' \cap X = \emptyset$.*

COROLLARY 5.12. *If any of $\tilde{\Gamma}_Q^S(X)$, $\tilde{\Gamma}^S(X)$ is a win for D, then every perfect set P has a perfect subset P' such that $P' \cap X = \emptyset$.*

By Theorem 5.5 and Remark 5.4, Theorem 5.11 is not true with ${}_{\mathbf{a}}\Gamma_Q^S$ or even with ${}_{\mathbf{a}}\Gamma^*$ instead of $\tilde{\Gamma}_Q^S$. It is an open problem whether it is true with ${}_{\mathbf{a}}\Gamma^S$. We conclude, however, from Theorem 4.7 and the fact that every perfect set has $2^{\aleph_0}$ perfect mutually disjoint subsets, that if ${}_{\mathbf{a}}\Gamma^{\smile}(X)$ is a win for D, then for every perfect P, $P - X$ has the power of the continuum.

A weakened form of Theorem 5.11 for ${}_{\mathbf{a}}\Gamma_Q^S(X)$ follows from Theorem 5.10.

COROLLARY 5.13. *Let* a *be a con. If* ${}_a\Gamma^S_Q(X)$ *is a win for D, then every nonempty open interval has a perfect subset which is disjoint from X.*

We summarize what we know of X given that ${}_a\Gamma^S_Q(X)$ or ${}_a\Gamma^S(X)$ is a win for D:

COROLLARY 5.14. *Let* a *be a con. If either of* ${}_a\Gamma^S(X)$, ${}_a\Gamma^S_Q(X)$ *is a win for D then X is of the first category and hence every nonempty interval has a perfect subset that is disjoint from X. If* ${}_a\Gamma^S(X)$ *is a win for D then, in addition, every perfect set has a subset of the power of the continuum that is disjoint from X.*

We mention that the last statement does not hold with ${}_a\Gamma^S_Q$, by Theorem 5.5.

COROLLARY 5.15. *If neither X nor* $R-X$ *include a perfect suvbset, then X is nondetermined in any of our games, an exception being* Γ^D *and* Γ^D_Q, *where X is a win for S.*

REFERENCES

1. A. Ehrenfeucht and G. Moran, *Size direction games over the Real Line. I*, Israel J. Math. **14** (1973), 163-168.

2. G. Moran, *Existence of non-determined sets for some two person games over reals*, Israel J. Math. **9** (1971), 316–329.

3. G. Moran and S. Shelah, *Size direction gamcs over the Real Line. III*, Israel J. Math. **14** (1973), 442–449.

4. R. Solovay, *A model of set theory in which every set is Lebesgue measurable*, Ann. of Math. **92** (1970), 1–56.

THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL
AND
UNIVERSITY OF FREIBURG
FREIBURG, GERMANY